PLUG EFFECT OF ERYTHROCYTES IN CAPILLARY BLOOD VESSELS

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ABSTRACT As an idealized problem of the motion of blood in small capillary blood vessels, the low Reynolds number flow of plasma (a newtonian fluid) in a circular cylindrical tube involving a series of circular disks is studied. It is assumed in this study that the suspended disks are equally spaced along the axis of the tube, and that their centers remain on the axis of the tube and that their faces are perpendicular to the tube axis. The inertial force of the fluid due to the convective acceleration is neglected on the basis of the smallness of the Reynolds number. The solution of the problem is derived for a quasi-steady flow involving infinitesimally thin disks. The numerical calculation is carried out for a set of different combinations of the interdisk distance and the ratio of the disk radius to the tube radius. The ratio of the velocity of the disk to the average velocity of the fluid is calculated. The different rates of transport of red blood cells and of plasma in capillary blood vessels are discussed. The average pressure gradient along the axis of the tube is computed, and the dependence of the effective viscosity of the blood on the hematocrit and the diameter of the capillary vessel is discussed.

INTRODUCTION

The capillary blood vessels are comparable in diameter with the red blood cells. The picture of the blood flow in the capillaries is that of a file of red blood cells suspended in the plasma. If the diameter of the capillary is considerably smaller than that of the red blood cell, the plasma is literally trapped between two consecutive red blood cells, and the bolus flow as described by Prothero and Burton (1961), and Lew and Fung (1968) develops. If the diameter of the capillary is larger than that of the red blood cell, the red blood cells would partially plug the flow of plasma. In the latter case there arise many questions such as "What is the ratio of the velocity of the red blood cell to the average velocity of the plasma?" and "How does the pressure drop change due to the plug effect of the red blood cells?" and so on.

In vivo observations of the blood flow in the capillaries show that the red blood cells take a great variety of shapes and positions. It is out of the question to analyze the mechanics of all configurations. Nevertheless, it would be useful to study an idealized model of such a flow.

Several idealized models have been analyzed. Wang and Skalak (1967) analyzed the flow in a cylindrical tube containing a line of spherical particles. Chen (1968) extended the analysis to a line of spheroids. Hyman and Skalak (1969) analyzed further the suspension of liquid drops in a cylindrical tube. Long bubbles in a tube have been analyzed by Bretherton (1961) Davies and Taylor (1949), Cox (1962), and Goldsmith and Mason (1962). A single spherical drop in an infinite tube was considered by Haberman and Sayre (1958). The work of Skalak and his coworkers demonstrated the importance of the gap between the particles and the tube wall on the drag of the particles. The effect of the neighboring particles is shown to be important only when the distance between the particles is sufficiently small.

In the present paper the idealized problem of a line of disks suspended in a cylindrical tube is analyzed. Our problem is mathematically simpler than those of Skalak and his coworkers. Our result gives the effects of the gap between the disks and the tube and of the interdisk distances. Our disks may be regarded as the limiting cases of the spheroids or cylindrical pillboxes; and our solution may serve as a standard with which the spheroidal or pillbox model results may be compared.

Of great interest are the ratio of the velocity of the disks to the mean velocity of the fluid, and the resistance to flow as affected by the disks. We shall show that both the velocity ratio and the resistance are affected by the gap between the disks and the tube and the distance between the disks. In this connection we shall point out that the solution for a line of closely spaced disks is qualitatively different from that of a floating, solid, cylindrical rod (enveloping the disks) moving in a cylindrical tube. The "rod" model has been used by a number of authors as an approximation to the blood flow in capillaries; but it is basically deficient in details and cannot serve as a satisfactory mathematical model for the problem at hand.

The suspended disks in our model are assumed not to tilt or shift so that their faces remains perpendicular to the tube axis and their centers remains on the axis. The flow will be axisymmetric because of the geometry of the boundaries. The Reynolds number of the plasma flow in the capillary blood vessel is of the order of 10^{-3}–1, in which range the effect of the inertial force is negligible compared to the pressure and friction forces, (see Lew and Fung, entry flow [0 and arbitraty], 1968). Consequently, the time-derivative term $(\partial \mathbf{v}/\partial t)$ in the Navier-Stokes equation can be neglected as well as the convective-inertia terms. Therefore, "creeping" flows are "quasi-steady". In other words the flow at any instant of time can be obtained from a steady-state solution of the Stokes equation compatible with the instantaneous boundary conditions. This basic fact will be used below to simplify the problem formulation.

GOVERNING EQUATIONS AND BOUNDARY CONDITIONS

The governing equations are the Navier-Stokes equation and the equation of continuity. The inertial-force term in the Navier-Stokes equation can be neglected because of the smallness of the Reynolds number; thus we have the Stokes equation

$$-\nabla p + \mu \nabla^2 \mathbf{v} = 0, \tag{1}$$

where p is the pressure, μ is the shear viscosity of the fluid, $\mathbf{v}$ is the velocity, and ∇ is the gradient operator. The fluid (plasma) is assumed to be incompressible. Thus, for the assumed axisymmetric flow the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0, \tag{2}$$

where r, θ, and x are a set of stationary cylindrical polar coordinates with the x axis coinciding with the axis of the tube and the origin of the coordinate system located on the middle plane between two adjacent disks at the moment of observation. The components of the velocity u and v are in the directions of x and r, respectively.

Because of the low Reynolds number and the neglect of the convective inertia, the flow can be assumed to be symmetric with respect to the middle plane between two adjacent disks, i.e.,

$$u(x, r) = u(-x, r) \tag{3}$$

$$v(x, r) = -v(-x, r). \tag{4}$$

This fact has been demonstrated in many well-known examples such as Stokes flow around spheres, cylinders, bolus flow, constricted tubes, etc., and can be verified in the present case *a posteriori*. Further, since the disks are assumed to be uniformly spaced, the flow has to be periodic. Hence, it is sufficient to formulate the problem in the right-hand side of the region between any two adjacent disks. The "no-slip" condition on the wall of the tube requires that

$$u(x, r)\,|_{r=a} = 0 \quad \text{for} \quad 0 \leq x \leq L \tag{5}$$

$$v(x, r)\,|_{r=a} = 0 \quad \text{for} \quad 0 \leq x \leq L, \tag{6}$$

where a is the radius of the tube, $2L$ is the distance between two adjacent disks, and U_{disk} is the velocity of the suspended disk relative to the tube wall. The boundary conditions on the plane of the disk are the adherence of the fluid to the surface of the disk and the periodicity of the flow; the former requires that

$$u(x, r)\,|_{x=L} = U_{disk} \quad \text{for} \quad 0 \leq r \leq b \tag{7}$$

$$v(x, r)\,|_{x=L} = 0 \quad \text{for} \quad 0 \leq r \leq b, \tag{8}$$

where b is the radius of the disk. On the annular region ($x = L, b \leq r \leq a$), which is an artificial boundary, the velocity components and their spatial derivatives must be continuous, i.e.,

$$\mathbf{v}(x, r)\,|_{x=L-} = \mathbf{v}(x, r)\,|_{x=L+} \quad \text{for} \quad b \leq r \leq a \tag{9}$$

$$\nabla\mathbf{v}(x, r)\,|_{x=L-} = \nabla\mathbf{v}(x, r)\,|_{x=L+} \quad \text{for} \quad b \leq r \leq a, \tag{10}$$

where $x = L-$ and $x = L+$ designate the planes defined by the left- and the right-hand side surface of the disc, respectively. The condition of equation 10 guarantees the continuity of the viscous stress on the annular plane ($x = L, b \leq r \leq a$). Thus, the continuity of the stress is satisfied if

$$p(x, r)\,|_{x=L-} = p(x, r)\,|_{x=L+} \quad \text{for} \quad b < r \leq a. \tag{11}$$

The balance of momentum of the fluid (Stokes equation) has to be satisfied in the annular region on the disk plane. Hence all terms involved in the Navier-Stokes equation must be continuous, i.e.,

$$\nabla^2\mathbf{v}(x, r)\,|_{x=L-} = \nabla^2\mathbf{v}(x, r)\,|_{x=L+} \quad \text{for} \quad b \leq r \leq a \tag{12}$$

$$\nabla p(x, r)|_{x=L-} = \nabla p(x, r)\,|_{x=L+} \quad \text{for} \quad b \leq r \leq a. \tag{13}$$

Our boundary-value problem is defined by equations 1–13. An additional condition is necessary to determine the constant U_{disk} (for example, the force acting on the disk). The problem belongs to the so called "mixed boundary condition" type, which normally is rather difficult to solve. In the following we shall solve this problem by a method similar to that employed by the authors in dealing with the entry flow and the bolus flow (Lew and Fung, 1969 *a*, 1969 *b*). It will be shown that no serious difficulty arises from the mixed boundary conditions.

METHOD OF ANALYSIS

We assume the following forms for $\mathbf{v}$, which satisfies the equation of continuity identically, and p:

$$\mathbf{v} = \nabla \times \nabla \times [\hat{x}f(x, r)] = -\hat{x}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial f}{\partial r} + \hat{r}\,\frac{\partial^2 f}{\partial r \partial x} \tag{14 a}$$

$$p = p_1 + \frac{\partial p_0}{\partial x}\,x\,. \tag{14 b}$$

Here $f(x, r)$ is a function of x and r, $\hat{x}$ and $\hat{r}$ are the unit base vectors of x and r axes, respectively, p_1 is the r-dependent portion of the pressure, and $(x)([\partial p_0]/[\partial x])$ represents that portion of the pressure which is independent of r. When equations 14 *a* and 14 *b* are substituted into equation 1, the following equation results:

$$-\nabla\left(p_1 - \mu\nabla^2 \frac{\partial f}{\partial x}\right) - \hat{x}\left(\frac{\partial p_0}{\partial x} + \mu\nabla^4 f\right) = 0, \qquad (15)$$

which is satisfied if

$$p = \mu\nabla^2 \frac{\partial f}{\partial x} + x \frac{\partial p_0}{\partial x}, \qquad (16)$$

and

$$\nabla^4 f = -\frac{\partial p_0}{\partial x}. \qquad (17)$$

An x-symmetric solution of equation 17 which has bounded values in $(-L \le x \le L, 0 \le r \le a)$ and can satisfy the zero-velocity condition on the wall of the tube, can be found as follows:

$$f(x, r) = a^2 U_{\text{dsk}} \sum_{n=1}^{\infty} A_n \cdot \left\{\left[-(1 + \lambda k_n \coth(\lambda k_n)) \frac{\cosh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} + k_n \frac{x}{a} \frac{\sinh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)}\right] \cdot \alpha_n \frac{J_0\left(k_n \frac{r}{a}\right)}{k_n^2} + B_{n0}\left(\frac{1}{8}\frac{r^4}{a^4} - \frac{1}{4}\frac{r^2}{a^2}\right) + \sum_{n=1}^{\infty} B_{nm} \frac{\cos\left(\frac{m\pi}{\lambda}\frac{x}{a}\right)}{(m\pi/\lambda)^2}\left[\beta_m I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right) + \gamma_m \frac{m\pi}{\lambda}\frac{r}{a} I_1\left(\frac{m\pi}{\lambda}\frac{r}{a}\right)\right]\right\}, \qquad (18)$$

where A_n, B_{nm}, α_n, β_m, γ_m, k_n, and m are arbitrary constants, J_i and I_i are the ith order Bessel function and the modified Bessel function, respectively, both of the first kind (see Hildebrand, 1962). λ designates the ratio of one-half of the inter-disk distance to the radius of the tube. The velocity U_{dsk} and the tube radius a are introduced to make the series on the right-hand side of equation 18 dimensionless. In selecting the solution above, v is made to vanish on the plane of the disk.

Substitution of equation 18 into equation 14 yields the following equations for the x and r components of the velocity:

$$\frac{u}{U_{\mathrm{dsk}}} = \sum_{n=1}^{\infty} A_n \left\{ \left[-(1 + \lambda k_n \coth(\lambda k_n)) \frac{\cosh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} + k_n \frac{x}{a} \frac{\sinh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} \right] \right.$$
$$\cdot\, \alpha_n J_0\left(k_n \frac{r}{a}\right) - B_{n0}\left(2\frac{r^2}{a^2} - 1\right) - \sum_{m=1}^{\infty} B_{nm} \cos\left(\frac{m\pi}{\lambda}\frac{x}{a}\right)\left[\beta_m I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right)\right.$$
$$\left.\left. + \gamma_m \left(2I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right) + \frac{m\pi}{\lambda}\frac{r}{a} I_1\left(\frac{m\pi}{\lambda}\frac{r}{a}\right)\right)\right]\right\}, \quad (19)$$

and

$$\frac{v}{U_{\mathrm{dsk}}} = \sum_{n=1}^{\infty} A_n \left\{ \left[\lambda k_n \coth(\lambda k_n) \frac{\sinh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} \right.\right.$$
$$\left. - k_n \frac{x}{a} \frac{\cosh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} \right] \alpha_n J_1\left(k_n \frac{r}{a}\right) - \sum_{m=1}^{\infty} B_{nm} \sin\left(\frac{m\pi}{\lambda}\frac{x}{a}\right)\left[\beta_m I_1\left(\frac{m\pi}{\lambda}\frac{x}{a}\right)\right.$$
$$\left.\left. + \gamma_m \frac{m\pi}{\lambda}\frac{r}{a} I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right)\right]\right\}. \quad (20)$$

The components of velocity satisfy the boundary conditions of equations 3, 4, and 8. An inspection of the boundary conditions on the wall of the tube, equations 5 and 6, suggests that there is more than one way to choose the constants so that the boundary condition of equations 5 and 6 are satisfied. We choose two sets of them as follows:

$$J_0(k_n) = 0,$$
$$\alpha_n = \frac{1}{J_1(k_n)},$$
$$\beta_m = \frac{1}{I_0\left(\frac{m\pi}{\lambda}\right)},$$
$$\gamma_m = -\frac{1}{2I_0\left(\frac{m\pi}{\lambda}\right) + \frac{m\pi}{\lambda} I_1\left(\frac{m\pi}{\lambda}\right)},$$
$$B_{n0} = 0,$$
$$B_{nm} = -\frac{4}{\lambda} k_n^2 \frac{\tanh(\lambda k_n)}{\left[k_n^2 + \left(\frac{m\pi}{\lambda}\right)^2\right]^2} \frac{\frac{m\pi}{\lambda}\cos(m\pi)}{\dfrac{I_1\left(\frac{m\pi}{\lambda}\right)}{I_0\left(\frac{m\pi}{\lambda}\right)} - \dfrac{\frac{m\pi}{\lambda} I_0\left(\frac{m\pi}{\lambda}\right)}{2I_0\left(\frac{m\pi}{\lambda}\right) + \frac{m\pi}{\lambda} I_1\left(\frac{m\pi}{\lambda}\right)}} \quad (21)$$

for solution 1, (f^1), and

$$J_1(k_n) = 0,$$

$$\alpha_n = \frac{1}{J_0(k_n)},$$

$$\beta_m = \frac{1}{I_1\left(\frac{m\pi}{\lambda}\right)},$$

$$\gamma_m = -\frac{1}{\frac{m\pi}{\lambda} I_0\left(\frac{m\pi}{\lambda}\right)},$$

$$B_{n0} = -\frac{2}{\lambda}\frac{1}{k_n}\tanh(\lambda k_n),,$$

$$B_{nm} = -\frac{2}{\lambda} k_n \tanh(\lambda k_n) \frac{\cos(m\pi)}{k_n^2 + \left(\frac{m\pi}{\lambda}\right)^2}\left[1 + \frac{k_n^2 - \left(\frac{m\pi}{\lambda}\right)^2}{k_n^2 + \left(\frac{m\pi}{\lambda}\right)^2}\right]$$

$$\cdot \frac{1}{\frac{I_0\left(\frac{m\pi}{\lambda}\right)}{I_1\left(\frac{m\pi}{\lambda}\right)} - \frac{2I_0\left(\frac{m\pi}{\lambda}\right) + \frac{m\pi}{\lambda} I_1\left(\frac{m\pi}{\lambda}\right)}{\frac{m\pi}{\lambda} I_0\left(\frac{m\pi}{\lambda}\right)}}, \quad (22)$$

for solution 2, (f^2).

In the case of solution 1, the condition $u = 0$ at $r = a$ is satisfied term by term, while the condition $v = 0$ at $r = a$ is satisfied by the Fourier sine series summed over m which determines the value of B_{n0} and B_{nm}. On the other hand the condition $v = 0$ at $r = a$ is satisfied term by term for solution 2, while the condition $u = 0$ at $r = a$ is satisfied by the Fourier cosine series expansion summed over m which determines the value of B_{n0} and B_{nm}.

Now, all constants are determined with the exception of A_n. The boundary conditions of equations 3–6 and 8 are satisfied by both solutions. Since u is symmetric with respect to the middle plane between two adjacent disks and $v = 0$ identically on the plane of the disk, the conditions of equations 9 and 10 are also satisfied by solutions 1 and 2.

Substitution of equation 18 into equation 16 yields the following equation for the pressure:

$$\frac{p}{\mu \frac{U_{\text{dsk}}}{a}} = 2\sum_{n=1}^{\infty} A_n \left\{ k_n \frac{\sinh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} \alpha_n J_0\left(k_n \frac{r}{a}\right) - 4B_{n0}\left(\frac{x}{a}\right) \right.$$

$$\left. - \sum_{m=1}^{\infty} B_{nm} \frac{m\pi}{\lambda} \sin\left(\frac{m\pi}{\lambda}\frac{x}{a}\right) \gamma_m I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right) \right\}. \quad (23)$$

Since p is antisymmetric, equation 13 is satisfied if equation 11 is satisfied. By inspecting the Stokes equation, we find that equation 12 is satisfied if equations 9 and 11 are satisfied. Therefore, only the boundary conditions of equations 7 and 11 need to be satisfied. This pair of the boundary conditions is a mixed type from which only undetermined constants, A_n's, can be determined.

Since p is antisymmetric with respect to the middle plane between two adjacent disks, the condition of equation 11 requires that p be a constant on the annular plane ($b < r \leq a$). This annular plane is an artificial boundary and no singularity in pressure should exist. Therefore, we choose the constant value of the pressure on this annular plane as the value of pressure on the wall of the tube, i.e.,

$$p(x, r)\,|_{x=L} = p(x, r)\,|_{x=L, r=a} \quad \text{for} \quad b < r \leq a. \quad (24)$$

When the conditions of equations 7 and 24 are imposed on equations 19 and 23, respectively, the following equations result:

$$\sum_{n=1}^{\infty} A_n \left\{ [-(1 + \lambda k_n \coth(\lambda k_n)) + \lambda k_n \tanh(\lambda k_n)] \alpha_n J_0\left(k_n \frac{r}{a}\right) \right.$$

$$- B_{n0}\left(2\frac{r^2}{a^2} - 1\right) - \sum_{n=1}^{\infty} B_{nm} \cos(m\pi) \left[\beta_m I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right) + \gamma_m \left(2I_0\left(\frac{m\pi}{\lambda}\frac{r}{a}\right) \right.\right.$$

$$\left.\left.\left. + \frac{m\pi}{\lambda}\frac{r}{a} I_1\left(\frac{m\pi}{\lambda}\frac{r}{a}\right)\right)\right]\right\} = 1 \quad \text{for} \quad 0 \leq r \leq b, \quad (25)$$

and

$$\sum_{n=1}^{\infty} A_n k_n \tanh(\lambda k_n) \alpha_n \left[J_0\left(k_n \frac{r}{a}\right) - J_0(k_n) \right] = 0 \quad \text{for} \quad b < r < a. \quad (26)$$

These equations define a mixed boundary condition on the plane of the disk from which the coefficients A_n must be determined. Only a limited analytical investigation has been done on this type of boundary conditions and no analytical method is available at the present time to determine A_n as far as the authors are aware.

Therefore, we shall use the collocation method which has been found to be quite effective in related entry and bolus flow problems. First, we apply equations 25 and 26 to a set of points distributed in the region $0 \leq r < a$; then, we solve the resulting set of simultaneous algebraic equations for A_n's involved in the truncated series solution.

When A_n's involved in the truncated series are determined, we can compute the velocity by equations 19 and 20. The pressure can be computed from equation 23. The pressure in the region to the right of the disk can be obtained by adding $2p(L, a)$ to equation 23. The mean pressure can be found from equation 23 as follows:

$$\frac{<p>}{\mu \frac{U_{\text{dsk}}}{a}} = 4 \sum_{n=1}^{\infty} A_n \left\{ \frac{\sinh\left(k_n \frac{x}{a}\right)}{\cosh(\lambda k_n)} \alpha_n J_1(k_n) - 2B_{n0}\left(\frac{x}{a}\right) - \sum_{m=1}^{\infty} B_{nm} \sin\left(\frac{m\pi}{\lambda}\frac{x}{a}\right) \gamma_m I_1\left(\frac{m\pi}{\lambda}\right) \right\}. \tag{27}$$

The equation for the mean speed of the flow can be easily obtained from equation 19 as follows:

$$\frac{<u>}{U_{\text{dsk}}} = 2 \sum_{n=1}^{\infty} A_n \left\{ [(1 + \lambda k_n \coth(\lambda k_n)) + \lambda k_n \tanh(\lambda k_n)]\alpha_n \frac{J_1(k_n)}{k_n} - \sum_{m=1}^{\infty} B_{nm} \frac{\cos(m\pi)}{m\pi/\lambda} \left[\beta_m I_1\left(\frac{m\pi}{\lambda}\right) + \gamma_m \frac{m\pi}{\lambda} I_0\left(\frac{m\pi}{\lambda}\right) \right] \right\}. \tag{28}$$

The force exerted on the disk by the stress in the fluid is given by the equation

$$F = 2\pi \int_0^b [p(x, r)|_{x=L-} - p(x, r)|_{x=L+}]r dr. \tag{29}$$

Since $p(x, r)|_{L+}$ is equal to $[p(x, r)|_{x=-L} + 2p(x, r)|_{x=L,r=a}]$, we find the equation for the force per unit area of the disk, $(\equiv F/\pi b^2)$, from equations 23 and 28 as follows:

$$\frac{f}{\mu \frac{U_{\text{dsk}}}{a}} = 8 \sum_{n=1}^{\infty} A_n \tanh(\lambda k_n)\alpha_n \left[\frac{J_1(\eta k_n)}{\eta} - \frac{k_n}{2} J_0(k_n) \right], \tag{30}$$

where η is the ratio of the disk radius to the radius of the tube.

From equations 21 and 23 we see that solution 1 represents a flow in which no net pressure drop along the axis of the tube exists. This is so because

$$p(x, r)|_{x=0} = 0 \quad \text{and} \quad p(x, r)|_{x=2L} = p(x, r)|_{x=0} + 2p(x, r)|_{x=L-,r=a},$$

which is equal to zero. On the other hand we see from equations 22 and 27 that the mean speed of flow is zero for the second solution, which corresponds to disks which move in the fluid but do not create a net mass flow. In other words solution 1 represents the fully developed flow induced by a series of moving disks through a tube connecting two tanks of the same pressure, while solution 2 corresponds to the fully developed flow induced by a series of moving disks through a tube connecting two sealed and filled tanks. Here the fully developed flow implies the flow far away from both ends of the tube, where neither the entry- nor the exit-flow effects exist. By combining these two independent solutions, we can obtain all different types of flow involving a series of disks. This theoretically completes the derivation of the solution of the problem under consideration.

NUMERICAL RESULTS AND DISCUSSION

The series involved in the derived solution is not orthogonal. Furthermore, the boundary conditions of equations 25 and 26 from which the unknown constants, A_n's, have to be determined are a mixed type instead of normal type. Since an analytical method which would enable one to determine the value of A_n is not available at the present time, we shall determine them by the collocation method.

In the calculation to be presented at the end of this paper, the series summed over

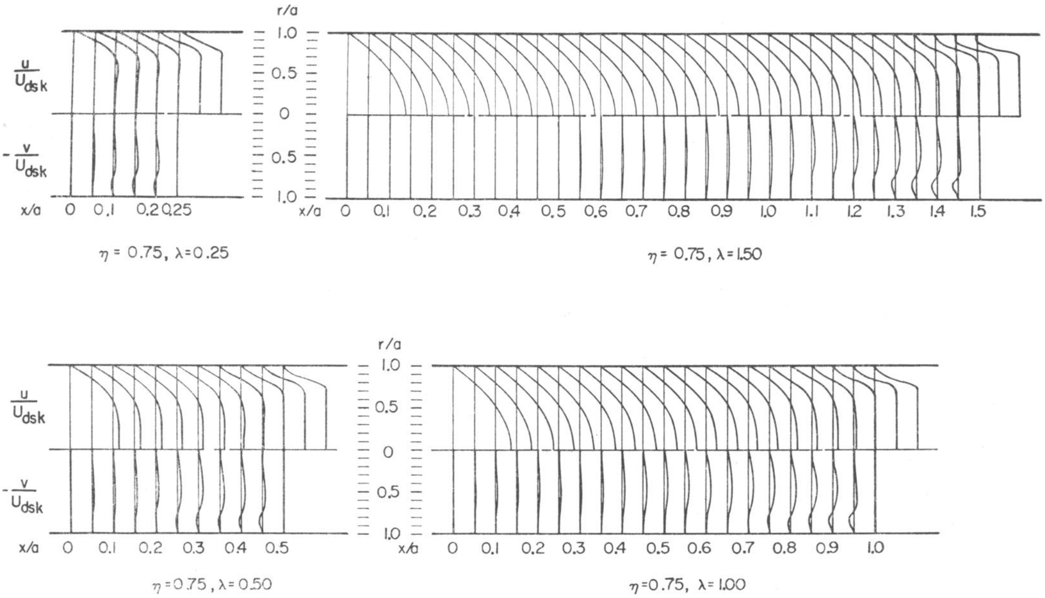

FIGURE 1 *a* Velocity distribution of the fluid of solution 1 in the right-hand side half of the region between two adjacent disks for $\eta = 0.75$. The figure shows in each cross-section the axial component of velocity u above the center line, and the radial component of velocity v below the center line. The parameter η is equal to (diameter of disks)/(tube diameter), and λ is equal to (distance between disks)/(tube diameter).

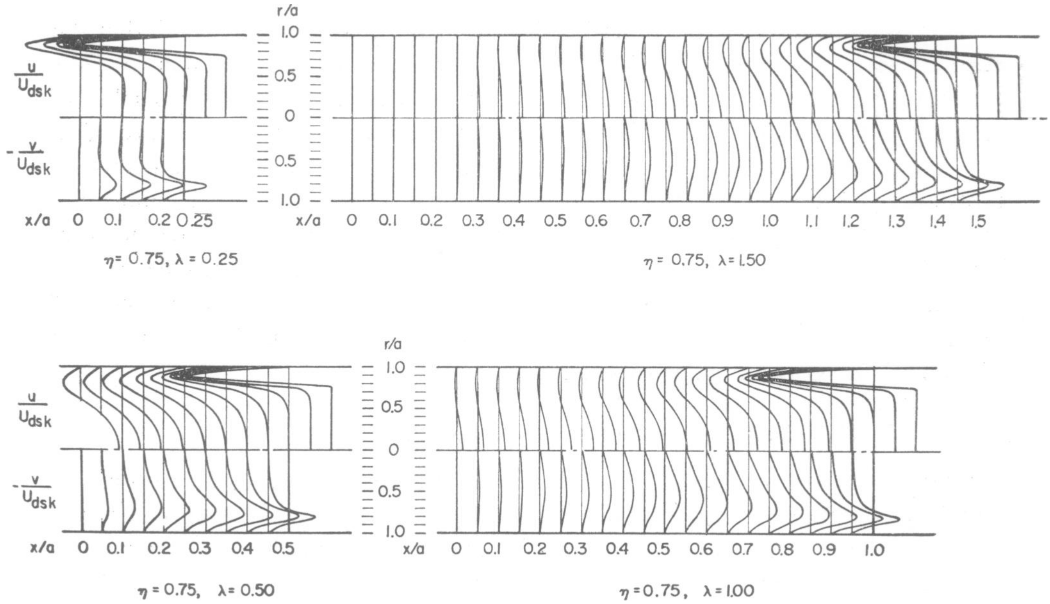

FIGURE 1 *b* Velocity distribution of the fluid of solution 2 for $\eta = 0.75$.

n is truncated by retaining the first 20 terms and the Fourier sine and cosine series expansion (series summed over m) are truncated after retaining the first 56 terms. Then, the boundary conditions of equations 25 and 26 are imposed at 20 different points which are uniformly distributed in the annular plane ($x = L$, $0 \leq [r/a] \leq 0.95$). The condition of equation 25 is imposed at those points in a closed interval $[0, \eta]$ and the condition of equation 26 is imposed at those points in a closed interval $[\eta + 0.05, 0.95]$. These provide 20 algebraic equations from which the 20 unknown constants involved in the truncated series solution are determined. In this computation 12 combinations of four different values of interdisk distance, $2\lambda = 0.5$, 1.0, 2.0, and 3.0, and three different values of the disk size, $\eta = 0.75$, 0.50, and 0.25, are considered. The magnitude of these computed values of A_n decreases quite rapidly as n increases more or less in a monotonical fashion. The ratio of the magnitude of the A_1 to A_{20} is of the order of 10^{-8} or smaller. This is comparable to the degree of convergence of the series involved in our investigations of the bolus and entry flow; these investigations involved ordinary instead of mixed boundary-value problems. From this fact we may conclude that the mixed boundary condition does not introduce any more difficulty in determining the unknown constants than the normal type of boundary condition, as long as the unknown constants are determined by the collocation method.

The distributions of the velocity are calculated from the truncated form of equations 19 and 20 and are shown in Figs. 1 *a–d*. As we can see in these figures, the perturbation of flow created by a disk becomes negligible (less than 1% of the

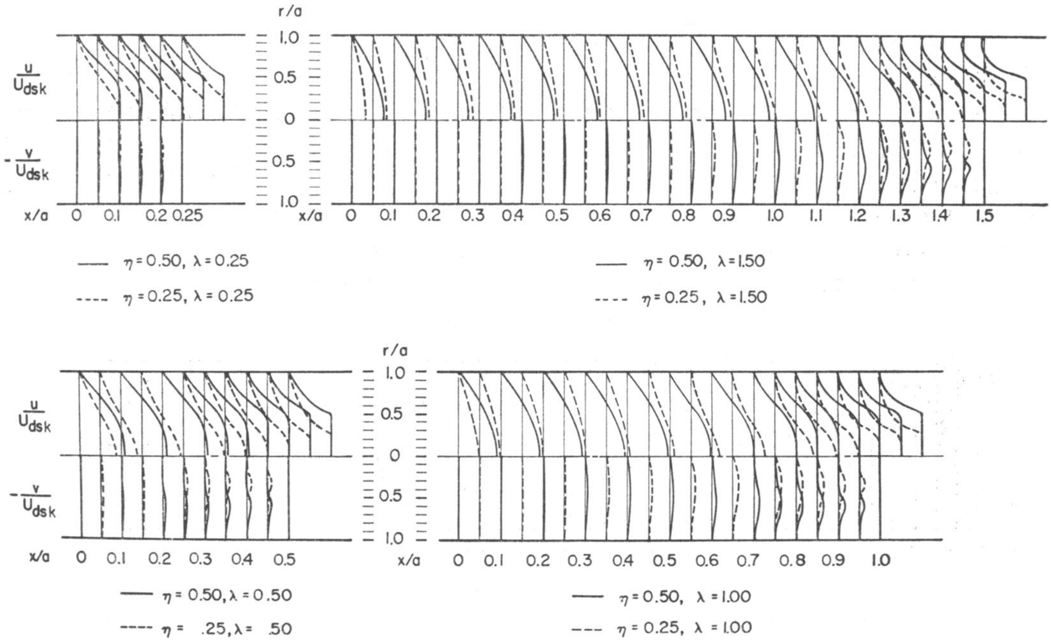

FIGURE 1 *c* Velocity distributions of the fluid of solution 1 for $\eta = 0.50$ and 0.25.

velocity of the disk) if the distance from the disk to the point of observation is equal to or greater than 1.3 times the radius of the tube. In other words the flow becomes the Poiseuille flow for solution 1 and it becomes null for solution 2. Therefore, we can say that two adjacent disks do not interact when they are separated by a distance equal to or greater than 2.6 times the radius of the tube. This fact was demonstrated in our earlier studies of the entry and bolus flow. It is also interesting to note that for solution 1 there is a small amount of back flow near the wall of the tube for a larger value of η. As will be shown, the discontinuous increase of the average pressure over the cross-section of the tube across the plane of the disk becomes larger and larger as λ increases. When λ becomes sufficiently large (e.g., $\lambda = 1.5$), this pressure jump becomes strong enough to cause such a back flow.

In Table I the mean speed of flow $<U>/U_{dsk}$, the mean pressure gradient $(\overline{[\partial\langle p\rangle]/[\partial x]})/\mu(U_{dsk}/a^2)$, the force $F/\pi\mu a U_{dsk}$, and the force densitw on the disk $f/\mu(U_{dsk}/a)$ are presented. The mean speed of flow is calculated by equation 28. From equation 27 and the preceding paragraph, we can easily see that

$$p(x, r)\,|_{x=0} = 0 \quad \text{and} \quad p(x, r)\,|_{x=2L} = 2p(x, r)\,|_{x=L,r=a}\,.$$

Hence, the mean pressure gradient can be found as

$$\overline{\frac{\partial<p>}{\partial x}} = p(x, r)\,|_{x=L,r=a}/L.$$

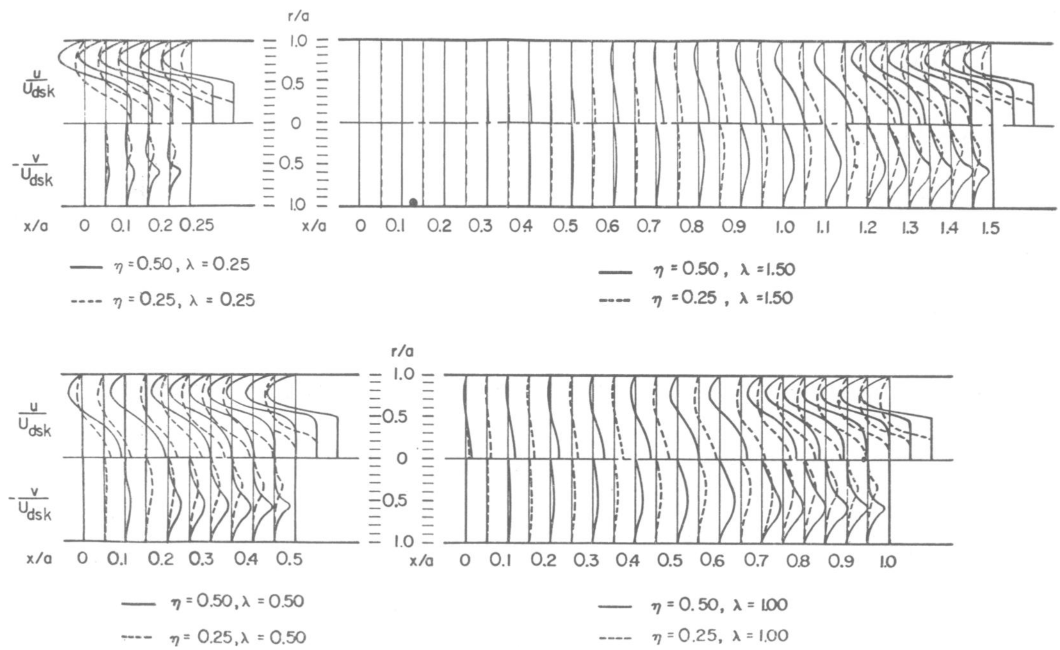

FIGURE 1 *d* Velocity distributions of the fluid of solution 2 for $\eta = 0.50$ and 0.25.

The force acting on the disk can be calculated by equation 29. In fact the evaluation of the integral involved in equation 29 is not necessary to find the force on the disk when the mean pressure over the cross-section of the tube is computed by equation 27. This is so because the force on the disk is equal to the difference between the mean pressure (average value over the cross-section of the tube) drop from $x = -L$ to $x = L$ and that from $x = 0$ to $x = 2L$. In this study we used the latter method. We mention once more that the net pressure drop for solution 1 is equal to zero and the net mass flow for solution 2 vanishes. These are the direct consequences of the selection of eigenvalues and eigenfunctions involved in the series solution and are regarded as a natural way to split arbitrary flow into two basic independent flows.

Now, we can obtain all types of solutions of interest to us by superimposing solutions 1 and 2 in a desired manner. In the blood flow through capillary blood vessels, flow is not always completely steady. Johnson and Wayland observed a rather significant fluctuation in the velocity of red blood cells over the period of observation (1967). It was also reported by those authors that the flow is comparatively constant in a few capillaries. The density of the red blood cell is comparable to the density of the plasma and the flow in the capillary blood vessel (the motion of plasma as well as that of red cells) is regarded as a low speed quasi-steady flow. Hence, the inertial force of the red blood cell has to be very small and, consequently, the force exerted on the red blood cell by the surrounding plasma also has to be

TABLE I
SUMMARY OF SOLUTIONS

η	λ	Solution 1					Solution 2				
		U_{dsk}	$< U >$	$\frac{\overline{\partial < p >/\partial x}}{\mu U_{dsk}/a^2}$	$\frac{F}{\pi\mu a U_{dsk}}$	$\frac{f}{\mu U_{dsk}/a}$	U_{dsk}	$< U >$	$\frac{\overline{\partial < p >/\partial x}}{\mu U_{dsk}/a^2}$	$\frac{F}{\pi\mu a U_{dsk}}$	$\frac{f}{\mu U_{dsk}/a}$
0.75	0.25	1	0.75844	0	−3.45688	−6.14556	1	0	249.62960	−109.70510	−195.03219
0.75	0.50	1	0.73456	0	−6.20520	−11.03147	1	0	151.10560	−117.08734	−208.15527
0.75	1.00	1	0.70668	0	−10.19520	−18.12480	1	0	75.89758	−117.43805	−208.77876
0.75	1.50	1	0.68200	0	−14.01880	−24.92231	1	0	50.59513	−117.43660	−208.77618
0.50	0.25	1	0.53438	0	−1.43096	−5.72384	1	0	32.68046	−9.59607	−38.30428
0.50	0.50	1	0.49865	0	−2.55752	−10.23008	1	0	22.02652	−13.54428	−54.17712
0.50	1.00	1	0.42691	0	−4.28344	−17.13376	1	0	11.48330	−14.07731	−56.30924
0.50	1.50	1	0.37193	0	−5.55920	−22.23680	1	0	7.65457	−14.07378	−56.29512
0.25	0.25	1	0.33534	0	−0.71480	−11.43680	1	0	7.27687	−1.93635	−31.41600
0.25	0.25	1	0.29210	0	−1.23896	−19.82336	1	0	5.21184	−2.77408	−44.38528
0.25	1.00	1	0.20919	0	−1.78568	−29.53088	1	0	2.76383	−2.92926	−46.86816
0.25	1.50	1	0.16086	0	−2.04448	−32.71168	1	0	1.84267	−2.92066	−46.73056

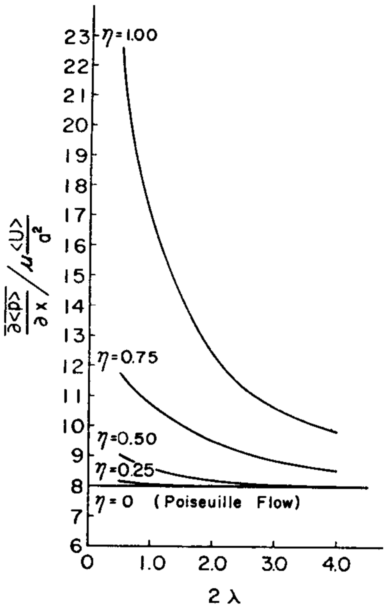

FIGURE 2 Average mean pressure gradient in the fluid as a function of η and λ for the flow involving a series of suspended disks.

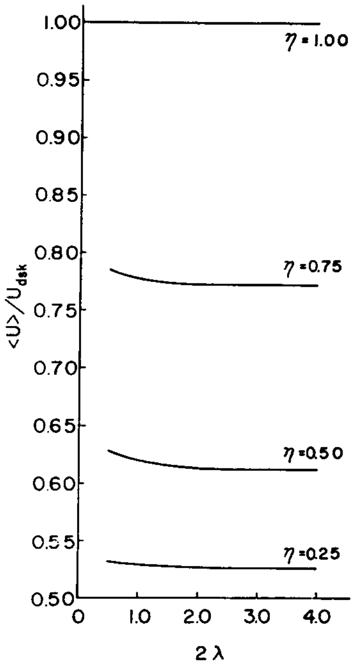

FIGURE 3 The ratio of the mean speed of the fluid to the velocity of the disks as a function of η and λ for the flow involving a series of suspended disks.

very small. This fact justifies the idea that the condition of zero force on the disk can be used as a criterion to superimpose two solutions for a steady flow as well as for a quasi-steady flow.

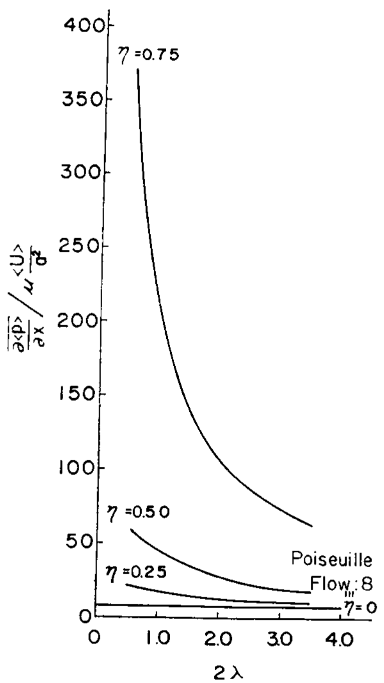

FIGURE 4 Average mean pressure gradient in the fluid in a tube involving a series of stationary discs as a function of η and λ.

When solutions 1 and 2 are superimposed in such a way that the force on the disk vanishes, ($F^1 + F^2 = 0$), we obtain the solution of a fluid flow involving a series of freely suspended disks. In Fig. 2 the average rates of mean pressure change along the axis of the tube are plotted as a function of the dimensionless radii of the disks and the dimensionless interdisk distance. The straight line of the averaged mean pressure gradient of 8 corresponds to the Poiseuille flow. When the relative disk size is small, the rate of mean pressure change is more or less equal to the Poiseuille flow for the moderate value of relative interdisk distance, but it deviates from the Poiseuille flow when the relative interdisk distance becomes very small. As the disk size increases, the average rate of mean pressure change increases rapidly. In other words the plug effect of disks becomes serious as the dimensionless radii of disks approach the value of unity. As mentioned at the beginning of this section, the computation is carried out for $\eta = 0.25$, 0.50, and 0.75 and $\lambda = 0.25$, 0.50, 1.00, and 1.50 in the present computation. The data corresponding to $\eta = 1$ are obtained from our previous study of the bolus flow (see Lew and Fung, 1969 *b*). We note that the rate of pressure change converges to that value of the Poiseuille flow if $\eta \rightarrow 0$ or $\lambda \rightarrow \infty$. In Fig. 3 the ratio of the mean speed of flow (of plasma) to the velocity of the disk (red blood cell) is plotted as a function of η (relative size of the red blood cell compared with the blood vessel) and $\lambda(1/[2\pi\lambda a^3]$ is equal to the number density of the disks or to the number of red blood cells per unit

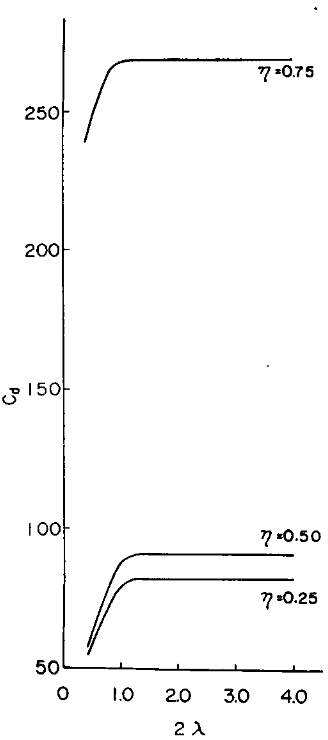

FIGURE 5 The drag coefficient of the stationary disks in the tube as a function of η and λ.

volume of the blood). It is interesting to note that the dependence of the ratio of the relative mean speed of the fluid to the velocity of disks on the relative interdisk distance is much weaker than in the case of pressure drop. We also note the large difference between the disk velocity and the mean fluid velocity.

In order to demonstrate another typical application of the derived set of solutions, the flow through a pipe involving a series of stationary discs is considered. This solution can be obtained by simply subtracting one solution from the other. In Fig. 4 the average rate of pressure change is presented. Here we notice the remarkable increase of the plug effect of stationary disks compared to the plug effect of suspended disks. The plug effect again vanishes if $\eta \rightarrow 0$ or $\lambda \rightarrow \infty$. In Fig. 5 the drag coefficient $C_d = F/(\mu a < U > \eta^2)$, where F is the drag force of the disk, is plotted versus the dimensionless interdisk distances by taking the dimensionless disk radius as a parameter. The dependence of C_d on η is very strong, in general. On the other hand its dependence on λ becomes insignificant when $\lambda \geq 2.0$.

CONCLUSIONS

The quasi-steady newtonian fluid flow involving a series of infinitely thin disks in a rigid, circular, cylindrical tube is studied by finding a series solution of the Stokes equation and the equation of continuity. In so doing it is assumed that the suspended disks are equally spaced along the axis of the tube, that their centers remain on the axis of the tube, and that their faces are perpendicular to the tube axis. The solution of the problem is split into two parts; solution 1 corresponding to a flow with zero net pressure drop and solution 2 corresponding to a flow with zero net mass flow. By superimposing these two fundamental solutions, all types of desired solutions can be obtained.

We obtained the solution for a flow involving a series of suspended disks by superimposing these two fundamental solutions in such a way that the forces on the disks in these two solutions cancel each other. The purpose of finding such a solution is to study a mechanism of plug effect of red blood cells in the blood flow through a capillary blood vessel. As mentioned in the Introduction of this paper, the simplest and yet realistic model of the blood flow in capillary blood vessels should be the flow involving a series of pillboxes instead of a series of disks. This would be particularly true for the skeletal muscle capillaries, the diameters of which are comparable with or even smaller than those of the red blood cells; the thickness of the severely deformed red blood cell thus becomes almost equal to the length of the plasma gap between adjacent red blood cells. This has to be so because the volume ratio of the plasma and the volume ratio of red blood cells are almost the same and because the plasma and the red blood cells are moving with the same velocity when the bolus flow develops. Although the pillbox-series model is a realistic one, it is a much more complicated model for the mathematical analysis compared with the disk-series model. Indeed, the disk-series model defines a rather well-defined boundary-value problem with a standard mixed boundary condition. Mathematically, the disk-series model serves our purpose as a stepping stone to the analysis of the pillbox-series model. Physically, this analysis of the disk-series model provides us with first hand information about the plug effect of red blood cells, the resulting pressure drop through the plasma gap between adjacent red blood cells (in order to obtain the total pressure drop, the pressure drop across the red blood cells has to be added), and the ratio of the mean plasma speed to the velocity of the red blood cells. Therefore, it is justifiable and useful to make some additional interpretation of this analysis in connection with the blood flow in the capillary blood vessels.

First of all, it is clear from Fig. 2 that the plug effect of red blood cells is very serious in the capillary blood vessels. This plug effect depends very strongly on η in a range of $1 \leq \eta \leq 0.50$ and λ in a range of $0 \leq \lambda \leq 3.0$. This proves the deficiency of the rod model, because such a model is used to simulate a series of very closely spaced red blood cells where the role of plasma gap between red blood cells becomes very important in the dynamics of the flow. The reason why the viscosity of blood depends on the size of tubes when the tube diameter is of the same order of magnitude as the red blood cells also becomes clear from Fig. 2. In fact we should use the rate of mean pressure change instead of the viscosity of blood in such a small tube or in the micro–blood vessels. Fig. 3 shows a significant difference between the mean speed of plasma and the velocity of the red blood cells. This clearly indicates that one should not use the observed velocity of the red blood cells as the mean plasma speed whenever some gap exists between the red blood cells and the wall of the capillary blood vessel. Since the velocity of the red blood cells differs significantly from the mean velocity of the plasma, it is interesting to discuss the two different values of hematocrit; one value is computed from the observation of the blood flow in a capillary blood vessel, and the other is found from the collection of blood emerging from the same blood vessel. Since we considered an infinitely thin disk, number density of the disks shall be considered instead of volume ratio. If the disk has finite volume, the volume ratio (hematocrit) can be obtained by multiplying the volume of a single disk by the number density. We shall call the number density of disks computed from observation of the flow the "apparent number density (apparent hematocrit)" and that found from the collection of medium the "effective number density (effective hematocrit)." Without any difficulty we see that the apparent number density for the flow under consideration is equal to $1/(2\pi\lambda a^3)$. On the other hand the effective number density is equal to $1/(2\pi\lambda a^3) \cdot (U_{\text{disk}}/\langle U\rangle)$. Therefore, the ratio of the apparent value to the effective value is equal to $\langle U\rangle/U_{\text{disk}}$ for number density as well as hematocrit. Fig. 3 shows how different those two hematocrit values can be. The effective hematocrit represents the relative transport of the red blood cells through the capillary blood vessels to the transport of plasma and, thus, is a kinematically meaningful value. On the other hand the apparent hematocrit represents the degree of plug effect and, thus, is a dynamically meaningful value.

The authors wish to express their appreciation to Mrs. Claudia Lowenstein for her help in numerical calculations.

This work is supported partially by the United States Public Health Service National Institute of Health under Grant No. USPHS HE 12494-01, partially by the National Science Foundation under Grant No. GK-1415, partially by the San Diego County Heart Association under a Fellowship granted in the name of the late Marjorie Siner, and partially by the National Institute of Health under Grant No. 12065.

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Received for publication 12 August 1969.

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